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LETTER TO THE EDITOR

Recurrence and ergodicity breaking in a Hamiltonian toy model

Damián H Zanette

Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche e Instituto Balseiro, 8400 Bariloche, Río Negro, Argentina

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Abstract. The properties of recurrence and ergodicity breaking in a discrete model that simulates the generic Hamiltonian motion are studied. These properties are respectively characterized by the distribution of orbit periods and the division in sectors of phase space. Despite its simplicity, the model can exhibit an intricate structure and, in fact, is able to mimic both regular and chaotic evolution. This complexity is also revealed in the appearance of power-law decays in the period distribution.

Poincaré's recurrence theorem, which states that a bounded Hamiltonian system returns systematically to an arbitrarily small neighbourhood of its initial condition, is probably the most general result now available on Hamiltonian dynamics [1]. Unfortunately, the proof of this theorem fails to predict the time that a given system takes to return close to its initial state. Except for some very simple cases, there is no general rule to calculate the recurrence time.

Historically, the recurrence theorem played a fundamental role in the discussion on the compatibility of the mechanical (microscopic) and statistical (macroscopic) description of Hamiltonian dynamics. The success of statistical mechanics is indeed grounded on the fact that, in most of the cases relevant to real systems, the recurrence time is suspected to be extremely long. The evaluation of this time is therefore closely related to the definition of the validity range for the statistical mechanics of Hamiltonian systems.

In this letter, the recurrence time of a very simple discrete Hamiltonian model is calculated—exactly, but by numerical means. The model is a discretized version of the Ehrenfests' wind-tree model [2], and simulates a gas whose particles collide against fixed scattering centres, similar to the Lorentz gas [3]. Despite its simplicity, the dynamics of this model shares several relevant features with the generic Hamiltonian motion. As a by-product, ergodicity breaking is also analysed.

The system evolves in discrete unitary time steps on an $N \times N$ square lattice of unitary spacing. It consists of P non-interacting particles which occupy the lattice nodes with no exclusion rule. Each of them moves with unitary speed in one of the four directions defined by the lattice. A particle preserves its velocity unless it encounters a site with a reflecting wall. In this case, it deviates to one of the two perpendicular directions. Each wall can therefore have two different orientations and reflects particles on its two faces. M walls are distributed at random on the lattice, with the same probability for the two orientations.

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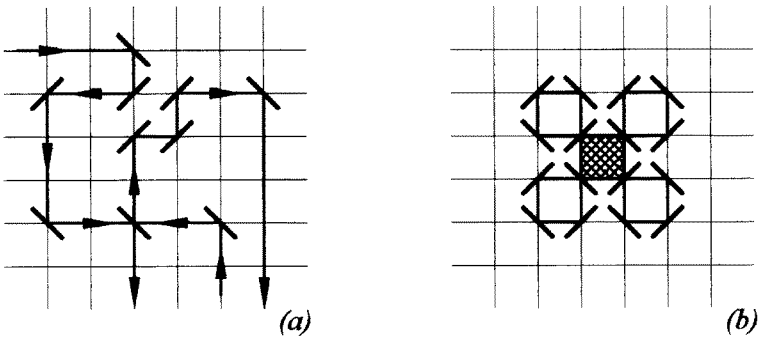

Figure 1. (*a*) Some trajectories of particles on the lattice. Note the two possible orientations of the walls, which reflect particles on both faces. (*b*) Four orbits of period $T = 4$ which determine a fifth one (broken).

This configuration is kept fixed along the whole evolution. Figure 1(*a*) shows a portion of the system, illustrating some of the possible orbits of a single particle.

Since the total number N_s of phase-space states is finite, $N_s = (4N^2)^P$, the system is necessarily recurrent [4], with a recurrence time $T \leqslant N_s$. Moreover, recurrence cannot occur to an intermediate state of the previous trajectory—giving rise to a limit cycle—but has to bring the system to its initial state. Returning to an intermediate state would in fact imply that such state has (at least) two predecessors, which is in contradiction with the reversible character of the dynamics. The model is therefore fully periodic and, thus, cannot exhibit the chaotic evolution typical of Hamiltonian systems [5]. As shown below, however, it can have orbits whose statistical properties mimic those of chaotic trajectories. Since the system consists of non-interacting particles, the case of a single particle, $P = 1$, will be considered first, discussing then the general case as a straightforward extension.

For a given configuration of the reflecting walls, the set of N_s phase-space states is divided in disjoint sectors, each corresponding to an orbit of the particle. From any initial condition in a given sector, the trajectory runs over all the other states before coming back to the initial point. The one-particle system is therefore not ergodic in a global sense, but ergodicity does hold inside each sector. This is analogous, in the motion of generic Hamiltonian systems, to the restriction of the trajectory to the constant energy manifold that contains the initial condition.

The division of phase space into sectors and the form of the trajectories are expected to become more and more complex as the number of reflecting walls increases. For $M = 0$ the phase space is divided in $s_0 = 4N$ sectors, all of them corresponding to trajectories of period $T = N$, which is thus the recurrence time of the system. In this case, the model mimics the Hamiltonian dynamics of a harmonic oscillator, where all the trajectories have the same period. For small values of M, $0 < M \ll N$, it can be shown that the number s of phase-space sectors is lower than s_0. A single reflecting wall ($M = 1$), for instance, couples the two orbits of period $T = N$ that intersect each other at the wall site, giving rise to a new orbit of period $T = 2N$. Accordingly, the number of sectors decreases to $s_1 = 4N - 2 < s_0$. On the other hand, for $M \approx N^2$, the division in sectors is mainly determined by short-period orbits. The number of phase-space sectors is then expected to be $s_{N^2} \propto N^2$. Hence, for sufficiently large N, $s_{N^2} \gg s_0$. This shows that the dependence of the number of sectors on the number of walls should exhibit two regimes. For small M, s decreases as M becomes larger, whereas for large M, s increases with growing M.

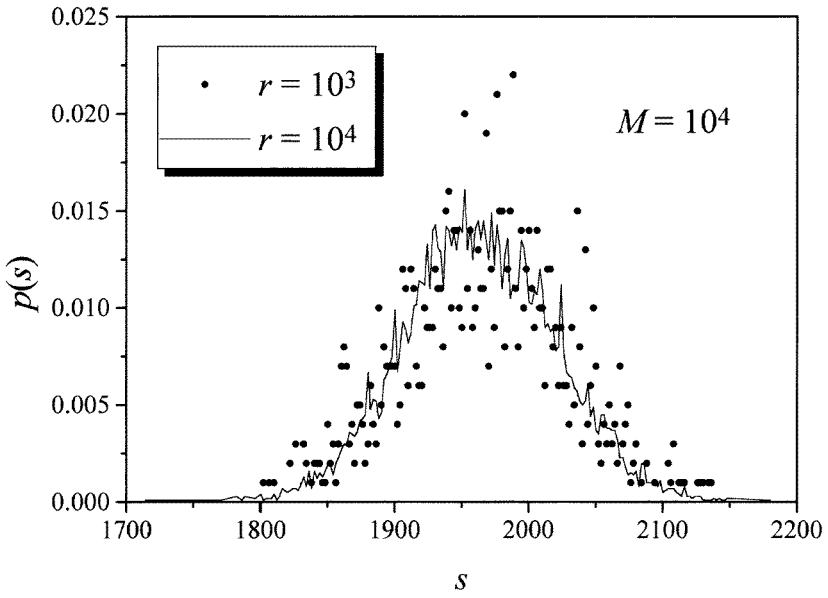

Figure 2. Normalized distribution of the number of sectors s, for $N = 100$ and $M = N^2$. The results correspond to averages over r realizations.

Note however that—since, for a given value of $M > 1$ there are several non-equivalent configurations of walls—s is not a well defined function of M. To characterize the typical values of s for fixed M it is then necessary to average over different realizations of the wall configuration. Figure 2 shows the normalized distribution of s, $p(s)$, at the maximum value $M = N^2$, numerically obtained for $r = 10^3$ and 10^4 realizations, and $N = 100$. This bell-shaped distribution, which is observed for all $M > 10$, suggests we characterize $p(s)$ by its mean value $\bar{s}$ and its quadratic dispersion $\delta s = \sqrt{\overline{(s - \bar{s})^2}}$. According to figure 2, these two quantities are well defined even by 10^3 realizations.

Figure 3 displays the results of numerical simulations over 10^3 realizations for $\bar{s}$ and δs, with $N = 100$. The two regimes predicted above are clearly seen, and the transition between them occurs at $M \approx 700$. The behaviour of δs is more complex, and the relative dispersion $\delta s/\bar{s}$—which is of the order 10^4 for small M—attains a maximum in the transition zone.

For this one-particle system, the recurrence time is simply given by the period of the orbit defined by the initial condition. For a given wall configuration, there is a certain distribution of periods which has in turn to be averaged over realizations, to produce a period distribution $p(T)$. This distribution, obtained from 10^4 realization, is shown in figure 4 for $N = 100$ and various values of M. The period distribution also shows two clearly defined regimes. For small M—precisely, in the region of decreasing $\bar{s}$ (cf figure 3)—$p(T)$ is strongly peaked, with sharp spikes at $M = nN$ ($n = 1, 2, \ldots$). The occurrence of these spikes can be explained noting that, as mentioned above, the presence of a wall couples two orbits of period N to give rise to an orbit of period $2N$, and so on. In the (log–log) plot for $M = 100$, it is clearly seen that the spikes decrease in amplitude following a power law in n. As M approaches the value at which the number of sectors is minimal, longer orbits should appear. Note in fact that for $M = 300$ a spike has appeared just below $T_{\max} = 2N^2$. In the region where $\bar{s}$ is minimum, then, practically maximal orbits occur with a rather high probability. As M grows further, the peaked distribution develops a background which,

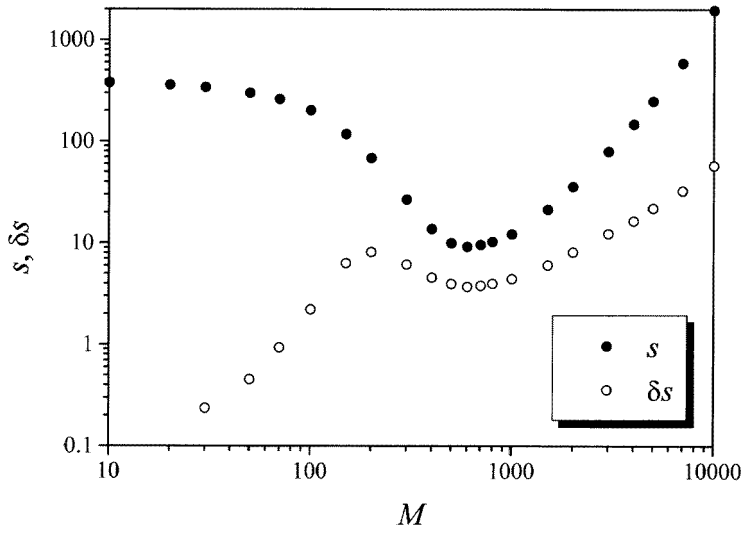

Figure 3. Mean number of sectors $\bar{s}$ and the corresponding quadratic dispersion δs as a function of M, for $N = 100$, averaged over 10^3 realizations.

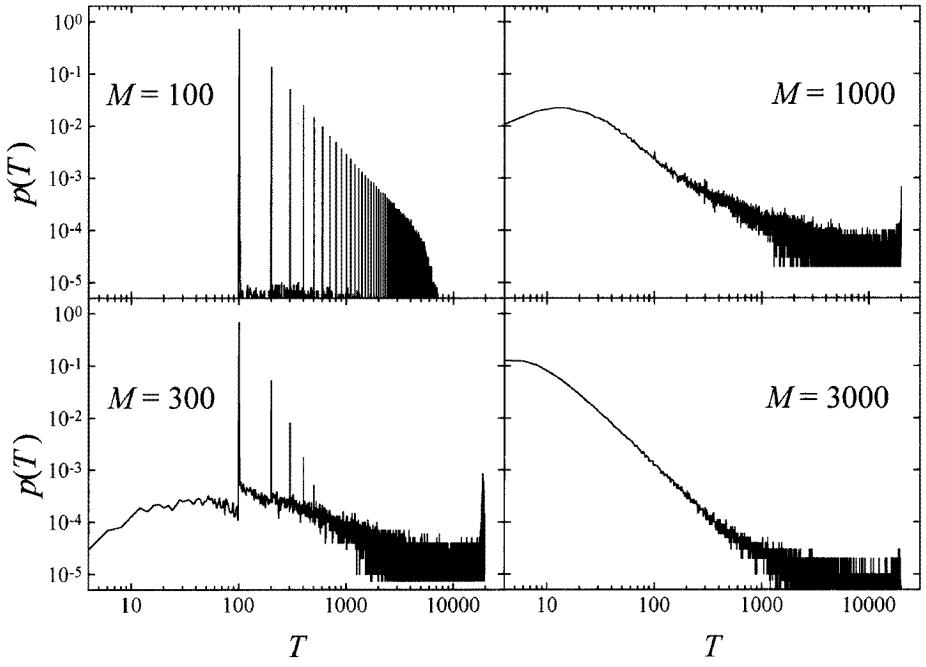

Figure 4. Normalized period distribution, $p(T)$, for $N = 100$ and several values of M, averaged over 10^4 realizations.

with respect to the spikes, can be considered as a 'continuous' distribution—although, of course, the values of T are discrete. For $M = 1000$, this smooth background dominates over the spikes, defining the second regime in the behaviour of $p(T)$. In this regime, the distribution shows a clear power-law decay. For the maximal value $M = N^2 = 10^4$ (not

shown in figure 4), $p(T) \propto T^{-\alpha}$ with $\alpha \approx 2.16$, over a wide interval.

The progressive division of phase space as more and more walls are added—which, for large M, produces an intricate structure of orbits—is a complex process driven by two competing effects. For small M, the process is dominated by the coallescence of relatively short orbits ($T \sim N$) into larger ones, and the number of sectors decreases. As M grows, long orbits are in turn fragmented in smaller sectors, whose number increases consequently. The formation of sectors can therefore be seen as a random sequence where, due to the branching process of coallescence and fragmentation, strong correlations appear (cf figure 1(*b*)). This coallescence-fragmentation mechanism—which is reminiscent of some well known physical processes that are able to produce power-law distributions [6]—could provide a clue to an explanation of the shape of $p(T)$.

Consider now the case of a many-particle system, $P \gg 1$. If in the initial condition the particles are uniformly distributed over the lattice and their velocities are chosen at random, the resulting distribution over sectors should be more or less uniform. The evolution of the system will then be nothing but a superposition of the periodic motions of the particles. The recurrence time will therefore be given by the least common multiple of the individual periods. This time is expected to be much shorter in the small-M regime, where all the periods are multiples of $T = N$, than in the large-M range, where many more values of T are admissible (cf figure 4).

The dynamics of the many-particle model mimics the evolution of a generic Hamiltonian system. On one hand, in fact, bounded integrable Hamiltonian systems can be reduced to a superposition of a discrete set of periodic motions [1, 5]. On the other, chaotic Hamiltonian systems are a superposition of periodic motions over a continuous set of periods [7]. Then, the model represents regular Hamiltonian dynamics for small M, where the period distribution is strongly peaked, and chaotic motion in the regime of large M, where the period distribution is 'continuous', in the sense discussed above. The two regimes observed in the behaviour of this model can thus be interpreted in terms of very generic properties of Hamiltonian systems.

From an alternative viewpoint, the model can be seen as a reversible inhomogeneous cellular automaton in phase space [8]. Topics such as ergodicity breaking and recurrence times have been studied in certain detail for other cellular automata, both abstractly and as models of real systems [4]. Some of the previous results—in particular, referring to period distributions [9]—are similar to the ones presented here. As a cellular automaton, the present model is a particularly intuitive representation of a Hamiltonian system. It would admit extension, for instance, to more dimensions, or to consider more complex collision laws. Remaining open problems, such as the origin of power-law distributions, provide a subject for further investigation.

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